Partial Differential Equations
Brownian Motion and Fourier's Heat Equation

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Brownian motion is named after Robert Brown, a Scottish botanist who discovered the phenomenon in 1827. He found that tiny particles suspended in a fluid medium have continual motion, making small irregular movements that are visible through a microscope (Kozdron, 2008). Almost a century later, it was shown that this phenomenon was due to the constant bombardment of the particulate molecules by the fluidic molecules of the medium (Einstein, 1926). After Browns first observations of this strange motion, it became a hot topic of research. It was researched by many different scientists, the most significant of which was Albert Einstein. Einstein wrote several famous papers on the topic of Brownian motion between the years 1905 and 1908. These papers provided the math necessary to satisfactorily describe this theory of erratic motion, which in turn gave a strong foundation for molecular and atomic theory. The following pages will cover the evolution of the conceptual theory of Brownian motion, along with an analysis of the mathematical representations of the theory, focusing on the satisfaction of Fourier’s famous heat/diffusion equation.

The History of the Theory

Conceptually Brownian motion has significant implications in predicting nature. It was found that when a great number of tiny particles were evenly dispersed in a fluid medium, they would create a “swarming motion” even when the gravitational force is neglected (Einstein [Fürth], 1926). Brown discovered this during his studies of plant pollen; for this study he had put pollen into a water medium to view it under a microscope. Brown found that even though the water was seemingly motionless the pollen particles in the water continued to move with constant erratic motions with no obvious forces at play. Since Browns original observation, this motion has been found to be a common property of both organic and inorganic particles of microscopic size, when emerged in a fluid medium; for example, smoke particles in air (Kozdron, 2008). Between the time of Brown and Einstein, many different scientists added their observations and comments to the body of analysis on Brownian motion.

The course of evolution of the theory of Brownian motion has taken several turns in the last hundred years. There are some hypotheses worthy of note that have helped to pave the path to the commonly accepted theory we have today. In 1857 Regnault hypothesized that Brownian motion was caused by inconsistent heating from incident light on the particles in the fluid. In 1865 Cantoni and Oehl found that the motion was continuous and unchanged when the particles and the fluid remained in a closed system for an entire year. In 1867 S. Exner found that the movement has an inverse relationship to the mass of the particles, the smaller the particles the more rapid the movement; he also found that the movement increased with an increase in heat or light. In 1870 Jevons thought that the movement was caused by electrical forces; but, this hypothesis was quickly disproven by Dancer that same year. Finally in 1877 Delsaux came to the commonly accepted conclusion that the erratic movements are caused by statistical collisions of the molecules of the fluid with the molecules of the particles (Einstein [Fürth], 1926). The statistical collision theory is the same mechanism Einstein used in his historical papers around thirty years later.

With the founding theory in place, innovative and defining experiments took place that furthered the explanation of Brownian motion. In 1888 Gouy made the first precise investigations to show that the motion increases with a lowering of the viscosity of the fluid medium. Gouy showed that there was no other relation to the intensity of light other than the excitation of the fluidic molecules which causes a decrease in viscosity. Gouy found that the motion of the particles was closely related to the thermal molecular motions of the fluid. He showed by measurement that the velocities of different particles themselves were around a hundred-millionth of their molecular velocity. In 1892 Ramsay described Brownian motion as a result of the pressure found in the fluid; this description was able to explain the departures from the established laws of osmotic pressure. In 1900 F.M. Exner was the last to add precise measurements to the theory before Einstein’s defining analysis in 1905. F.M. Exner established the theory that the velocities of the particles are decreased with an increase in the mass of the particles and that the velocities increase with an increase in temperature (Einstein [Fürth], 1926). The work done by Gouy and F.M. Exner were the foundation for Einstein's original thoughts on Brownian motion. Einstein used these first few precise measurements and calculations as the basis for his mathematically defining papers.

Einstein's Analysis

In 1905 Albert Einstein showed that the probability density function for the Brownian motion of particles satisfies Fourier's heat/diffusion equation (Sattinger, 2003). This significant discovery gave Brownian motion the mathematical representation needed to agree with pre-established thermodynamic theories of molecular motion. Einstein's derivation of the heat/diffusion equation with Brownian motion in 1905 has been said to be of equal significance as the other two papers he published that year. Einstein seemed to be very busy and productive in 1905, at the ripe age of 26; he wrote founding papers on relativity, Brownian motion, and the photoelectric effect which won him the Noble prize (Kozdron, 2008). Einstein's explanation of Brownian motion enticed a scientific revolution that was just as important as the relativistic or quantum revolutions. Brownian motion continues to be of immeasurable importance in modern science today, amplified in the fields of physics, biology and nanotechnology (Haw, 2005). It is a shame that this important theory has not received the focus that it deserves.

Einstein's derivation of Fourier's heat/diffusion equation using Brownian motion took the mathematical path that follows. Einstein began with a probabilistic model of a collision between the fluidic molecules of a medium with the particulate molecules that experience the micro-visible motion. Each collision has a probability that is independent of other collisions. There is always an equal probability of a collision, independent of how many collisions have previously occurred (Sattinger, 2003). Einstein asserted that the continual bombardment of the particulate molecules by the fluidic molecules causes the phenomenon of Brownian motion. The erratic motion is created by the differing incident angles of the fluidic molecules on the particulate molecules. This inconsistency causes the fluidic molecules to transfer differing magnitudes of impulse to the particulate molecules. As a result of these continual collisions, the particles attain the same average kinetic energy as the molecules.

The Math

Suppose there are n particles suspended in a fluid. In a short time interval τ, the x-coordinate of a single particle will increase by ε, where ε has a different value (positive or negative) for each particle $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_n)$. For the value of ε a certain probability law will hold; in the short time interval τ, the number dn of particles which experience a displacement between ε and $\varepsilon + \Delta\varepsilon$ can be expressed by the equation,

$$dn = n\,\varphi(\varepsilon)\,d\varepsilon,$$

where φ only differs from zero for very small values of ε and satisfies,

$$\int_{-\infty}^{\infty} \varphi(\varepsilon)d\varepsilon = 1 \qquad and \qquad \varphi(\varepsilon) = \varphi(-\varepsilon).$$

Since φ is an even function, we see that,

$$\int_{-\infty}^{\infty} \varepsilon\,\varphi(\varepsilon)d\varepsilon = 0.$$

Investigating how the coefficient of diffusion (i.e., thermal conductivity) depends on φ, shows that the number v of the particles per unit volume is only dependent on x and t. Indicating number of particles per unit volume, $v = f(x,t)$, we will calculate the distribution of the particles at a time $t + \tau$ from the distribution at the time t. From the definition of the function $\varphi(\varepsilon)$, the number of particles which are located at the time $t + \tau$ between two planes perpendicular to the x-axis, with abscissa $x\ and\ x + dx$,

$$f(x, t+\tau)dx = dn, \qquad and \qquad \int_{-\infty}^{\infty} \varepsilon\,\varphi(\varepsilon) = 0.$$

Let $f(x,t)$ denote the number of particles per unit volume at location x at time t so that,

$$\int_{-\infty}^{\infty} f(x,t)dx = n.$$

α^2 is defined by,

$$\alpha^2 = \frac{1}{2\tau}\int_{-\infty}^{\infty} \varepsilon^2 \varphi(\varepsilon) d\varepsilon.$$

The goal is to now calculate the distribution of particles a short time later. The definition of $\varphi(\varepsilon)$ gives,

$$f(x, t+\tau) = \int_{-\infty}^{\infty} f(x+\varepsilon, t)\ \varphi(\varepsilon) d\varepsilon. \qquad (1)$$

Since τ is very small, Taylor's theorem becomes simple,

$$f(x, t+\tau) = f(x,t) + \tau \frac{df}{dt}, \qquad (2)$$

this can be further expanded to include $f(x+\varepsilon, t)$ in powers of ε,

$$f(x+\varepsilon, t) = f(x,t) + \varepsilon \frac{\partial f(x,t)}{\partial x} + \frac{\varepsilon^2}{2!}\frac{\partial^2 f(x,t)}{\partial x^2} + \cdots \frac{\varepsilon^n}{n!}\frac{\partial^n f(x,t)}{\partial x^n}. \qquad (3)$$

The expansion can now be expressed as an integral, because only very small values of ε contribute anything to the equation, so it gives,

$$f + \frac{\partial f}{\partial t}\tau = f\int_{-\infty}^{\infty} \varphi(\varepsilon) d\varepsilon + \frac{\partial x}{\partial f}\int_{-\infty}^{\infty} \varepsilon\varphi(\varepsilon) d\varepsilon + \frac{\partial^2 f}{\partial x^2}\int_{-\infty}^{\infty}\frac{\varepsilon^2}{2}\varphi(\varepsilon) d\varepsilon + \cdots \frac{\partial^n f}{\partial x^n}\int_{-\infty}^{\infty}\frac{\varepsilon^n}{n!}\varphi(\varepsilon) d\varepsilon.$$

On the right-hand side, the second, fourth, sixth, etc., terms vanish since $\varphi(x) = \varphi(-x)$ (phi is an even function); while the first, third, fifth, etc., terms and every succeeding term is very small compared to its predecessors. Keep in mind that $\int_{-\infty}^{\infty} \varphi(\varepsilon) d\varepsilon = 1$ and that $\alpha^2 = \frac{1}{2\tau}\int_{-\infty}^{\infty} \varepsilon^2 \varphi(\varepsilon) d\varepsilon$. Now, insert equation (3) into equation (1) and it yields,

$$\begin{aligned} f(x, t+\tau) &= \int_{-\infty}^{\infty} f(x+\varepsilon, t) + \varphi(\varepsilon) d\varepsilon \\ &= \int_{-\infty}^{\infty} \left[f(x,t) + \varepsilon\frac{\partial f}{\partial x} + \frac{\varepsilon^2}{2}\frac{\partial^2 f}{\partial x^2}\right]\varphi(\varepsilon) d\varepsilon \\ &= f(x,t) + \alpha^2 \frac{\partial^2 f}{\partial x^2}\tau, \qquad (4) \end{aligned}$$

by using the properties of $\varphi(\varepsilon)$ above.

By equating the two approximation, one with respect to time (2), and the other with respect to (random) displacements (4), we obtain the partial differential equation,

$$\frac{\partial f}{\partial t} = \alpha^2 \frac{\partial^2 f}{\partial x^2}.$$

Which, if you do some slight rearranging, is the well-known differential equation for heat/diffusion concluded by Fourier,

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}.$$

From this satisfaction of the heat/diffusion equation, it may be concluded that,

$$f(x,t) = \frac{n}{\alpha\sqrt{4\pi t}} e^{-x^2/4\alpha^2 t}.$$

Notice that the formula for $f(x,t)$ is just n times a density function, as stated in the first paragraph of “Einstein’s Analysis”.

(References for entire “Math” section: Einstein, 1926 and Kozdron, 2008.)

Conclusions

The standard heat/diffusion equation is a second order partial differential equation. It has a second-order partial space-derivative of temperature on the left-hand side, which equals the inverse diffusion coefficient, or inverse thermal conductivity coefficient, multiplied by a first-order partial time-derivative of temperature on the right-hand side. The fact that Brownian motion satisfies the heat/diffusion equation has significant implications in a diverse group of sciences. It provides a solid foundation for molecular and atomic theory. These theories were not widely accepted during the time Einstein published his papers on Brownian motion. This lack of acceptance is the most likely reason that Einstein’s essays on Brownian motion have not received the attention or acknowledgement they deserve. Even today these essays are not commonly known in the elementary scientific community, though they are the most commonly cited of Einstein’s 1905 papers (Haw, 2005). The citation statistics show that Brownian motion is very important, or even necessary for the functioning of modern science. These statistics also provide hope that the theory of Brownian motion will be common scientific knowledge in the relatively near future.

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